PAULI MATRICES AND GELL-MANN MATRICES

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Abstract. I explain the difference between chalk and cheese, and attempt to turn water into wine.

1. Chalk

The Pauli matrices according to my favourite textbook are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

that is, the Hermitian matrices in physicists' convention. This is what they write on the blackboard, apparently, so it must be chalk.

2. Cheese

The anti-Hermitian matrices are obtained by multiplying by i to get the mathematicians' convention

$$k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{2}$$

in which the matrices k, j, i behave like quaternions $ij = -ji = k$ etc. There is no need to write this anywhere, just let it mature for a while, then take it out of a quart pot and taste it. It melts in the mouth, so it must be cheese.

3. Water

The Gell-Mann matrices are basically the same thing, inserted into the 2×2 blocks in a 3×3 matrix, but that gives you 9 matrices instead of 8, so some kludge is put in to make them orthonormal:

$$\begin{gathered}
\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{gathered} \tag{3}$$

This is what they use to describe the colours of quantum chromodynamics, but it has no flavour to it at all. It is just coloured water.

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4. Wine

The finite group of order 27 uses the complex numbers $v, w = (-1 \pm \sqrt{-3})/2)$. It doesn't matter which is v and which is w, you can choose whichever convention you like. The 27 matrices are these 9 multiplied by the three scalars $1, v, w$:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\0&v&0\\0&0&w\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\0&w&0\\0&0&v\end{pmatrix},$$
$$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\0&0&v\\w&0&0\end{pmatrix}, \quad \begin{pmatrix}0&1&0\\0&0&w\\v&0&0\end{pmatrix},$$
$$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\v&0&0\\0&w&0\end{pmatrix}, \quad \begin{pmatrix}0&0&1\\w&0&0\\0&v&0\end{pmatrix}. \tag{4}$$

Now throw away the first one, which is the only one that does not have trace zero, and ignore the scalar multiplies by v, w, and you have the 8 matrices you need. They are already orthonormal, so no kludge is required.

Can't you just taste the difference? Three bottles each of white, red and rosé. All we need now is three bottles of neutrino champagne to complete the case. That should really get the party going!