A DISCRETE MODEL FOR GELL-MANN MATRICES

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Abstract. I propose a discrete model for the Gell-Mann matrices, which allows them to participate in discrete symmetries of three generations of four types of elementary fermions, in addition to their usual role in describing a continuous group $SU(3)$ of colour symmetries. This model sheds new light on the mathematical (rather than physical) necessity for 'mixing' between the various gauge groups $SU(3)$, $SU(2)$ and $U(1)$ of the Standard Model.

1. Introduction

1.1. **Context, aims and objectives.** The Gell-Mann matrices [1, 2] are a particular choice of orthonormal basis for the (complex) Lie algebra $\mathfrak{su}(3)$, analogous to the Pauli matrices which form a basis for $\mathfrak{su}(2)$. They are an essential part of the calculational tools of quantum chromodynamics (QCD) as a description of the strong nuclear force [3, 4]. The analogy with Pauli matrices is not complete, however, since the Pauli matrices are unitary, but the Gell-Mann matrices are not. In particular, the Pauli matrices are non-singular, so they can also be used to describe a *finite* group of discrete (unitary) symmetries. This is useful for describing the discrete symmetries of weak isospin, that distinguishes electrons from neutrinos.

Specifically, the Pauli matrices generate a commuting product of the cyclic group Z_4 of order 4 with the quaternion group Q_8 of order 8, giving a finite analogue of the gauge group $U(1)_Y \times SU(2)_L$ of weak hypercharge and weak isospin. But the 'mixing' of weak hypercharge and weak isospin to create electric charge implies that $U(1)_{em}$ does not commute with $SU(2)_L$. The only reasonable way to accommodate this property in the finite groups is to take a cyclic group Z_3 in place of $U(1)_{em}$, and let Z_3 act as automorphisms of Q_8. This leads to a group of order 24, known as the binary tetrahedral group (among many other names), which is a *semi-direct* product $Q_8 \rtimes Z_3$ of Q_8 and Z_3, rather than a direct product $Q_8 \times Z_3$. Possible applications of this finite group to the modelling of elementary particles, and in particular of electro-weak mixing, are explored in [5, 6, 7, 8, 9, 10].

The Gell-Mann matrices, by contrast, are singular, so they do not generate a finite group, and they cannot be used for discrete symmetries such as the three-generation symmetry of electrons. This means that the three generations have to be added to the Standard Model 'by hand', rather than arising automatically from the formalism. The aim of this paper is to describe a method by which the Gell-Mann matrices can be given a discrete structure that might be useful for this purpose. I also explore options for mixing the strong force with the electro-weak forces by the process of using semi-direct products of the finite groups in place of the direct products of Lie groups. It turns out that there is a unique possibility, arising from a particular three-dimensional complex reflection group [11, 12] of order 648.

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Indeed, it is possible to extend this process one stage further, to include the Dirac matrices [13]. In a continuous model, this is not possible, as the Coleman–Mandula Theorem [14] tells us. But this theorem only applies to models based on Lie groups, and does not apply to models based on finite groups. I shall show that there is a unique possible action of the proposed finite analogue of the gauge group of the Standard Model on an appropriate finite analogue of the Dirac matrices. This requires extending the Dirac matrices from complex to quaternionic, in order to incorporate three generations of fermions, and extends the usual group of order 64 to order 128. The resulting 'unification' of Dirac, Gell-Mann and Pauli matrices into a single structure forms a group of order 82944, consisting of 4×4 quaternion matrices, that is a subgroup of a quaternionic reflection group [15, 16].

If this group is useful for physics, then it certainly goes beyond the Standard Model of Particle Physics, because it allows the Gell-Mann matrices and the Pauli matrices to act on the Dirac matrices, instead of commuting with them. In other words, it allows particle physics to determine the shape of spacetime, and possibly thereby to include a quantum theory of gravity. The main aim of this paper, however, is to explain the representation theory of the group of order 648 in relation to the Standard Model, rather than speculating about quantum gravity.

An important part of this aim is to understand how the action of Pauli matrices on the extended (quaternionic) Dirac matrices restricts to the Standard Model implementation of electro-weak mixing on the ordinary (complex) Dirac matrices. Similarly, the action of the Pauli matrices on the (modified) Gell-Mann matrices needs to be understood in the context of weak-strong mixing. In all cases, the finite group can give only the combinatorial structure of this mixing, without numerical values. The numerical values can only arise from choosing coordinates for the representations of the group, about which this model says nothing.

1.2. **Pauli matrices.** The standard textbook Pauli matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

that is, the Hermitian matrices in physicists' convention. These give rise to elements of $SU(2)$ via complex exponentiation as follows:

$$\begin{aligned} \exp(i\sigma_x\theta) &= \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \quad \exp(i\sigma_y\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \\ \exp(i\sigma_z\theta) &= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}. \end{aligned} \tag{2}$$

In addition there is a scalar matrix $i = \sigma_x\sigma_y\sigma_z$ which exponentiates to

$$\exp(i\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \tag{3}$$

as elements of a scalar group $U(1)$. If this copy of $SU(2)$ is used for weak isospin, then the corresponding copy of $U(1)$ is used for weak hypercharge.

Mathematicians prefer to put the multiplication by i into the definition of the matrices, instead of the exponentiation, so using the anti-Hermitian matrices

$$K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{4}$$

in which the matrices K, J, I behave like quaternions $IJ = -JI = K$ etc.

All the Pauli matrices, whether Hermitian or anti-Hermitian, are also unitary, as are the scalars. Therefore they generate finite subgroups of $U(2)$: in place of $U(1)_Y$ we have a cyclic group Z_4 of order 4 generated by the scalar i, and in place of $SU(2)_L$ we have a quaternion group Q_8 of order 8, generated by $i\sigma_x$, $i\sigma_y$ and $i\sigma_z$. In other words, there is a third interpretation of the Pauli matrices, as generators for unitary representations of a finite group, as well as the conventional interpretations as generators for unitary groups and Lie algebras. Such an interpretation is useful in cases (such as weak isospin) in which the underlying symmetry is discrete.

1.3. **A 3-dimensional analogue.** The Gell-Mann matrices in physicists' convention are 3×3 analogues of the Hermitian traceless Pauli matrices:

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{5}$$

Again, the mathematicians' convention is to multiply by i to get anti-Hermitian matrices, which can then be exponentiated to get unitary matrices generating $SU(3)$. In both cases, the Gell-Mann matrices form a basis for the 8-space of complex 3×3 traceless matrices.

There are other possible bases, and it is suggested in [17] that a basis of *real* matrices might be more fundamental, for mathematical rather than physical reasons. But in none of these conventions are the matrices themselves unitary, so that the finite symmetries one might hope to see (such as the generation symmetry for fundamental fermions) are not directly available in the standard formalism. In fact it is easy to write down an orthonormal basis of unitary matrices, as I shall now demonstrate.

The natural analogue in three dimensions of the quaternion group (of order $2^3 = 8$) in two dimensions is the group G_{27} of order $3^3 = 27$ generated by the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \exp(2\pi i/3) & 0 \\ 0 & 0 & \exp(4\pi i/3) \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{6}$$

Writing $v = \exp(2\pi i/3)$ and $w = v^2 = \exp(4\pi i/3)$ for simplicity, the trace zero matrices form a complex 8-space with the following orthonormal basis:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & w \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & v \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & v \\ w & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & w \\ v & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ v & 0 & 0 \\ 0 & w & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ w & 0 & 0 \\ 0 & v & 0 \end{pmatrix}. \tag{7}$$

The first four are the generators (6) and their inverses, and the last four are the products of the first two with the next two. The generators (of order 3) do not commute with each other, and the commutators $x^{-1}y^{-1}xy$ are scalar matrices of order 3. In other words, there is a scalar group of order 3 hidden in this group, consisting of scalars $1, v, w$, so that when we exponentiate these matrices there is a threefold ambiguity in the overall phase, corresponding to the fact that the Lie group $SU(3)$ has a centre consisting of scalars of order 3, The above matrices are all unitary, and all have determinant 1, so that the group of order 27 that they generate is a finite subgroup of $SU(3)$.

Moreover, these 8 matrices span the same complex 8-space as the 8 Gell-Mann matrices, namely the space of all complex matrices with trace zero. I therefore suggest that using these 8 matrices in place of the 8 Gell-Mann matrices makes essentially no difference to the Standard Model, as it is just a change of basis on the space of gluons. But it has the huge advantage over the Gell-Mann matrices, of incorporating finite (triplet) symmetries in a natural way. Indeed, it contains two, or even three, *independent* triplet symmetries, and may therefore be able to accommodate generation symmetries as well as colour (and anti-colour) symmetries.

1.4. **A 4-dimensional analogue.** The Dirac matrices are 4×4 complex matrices that have a similar structure to that of the Pauli matrices and Gell-Mann matrices. But in this case both Hermitian and anti-Hermitian matrices are required, and the group that is normally used is $SL(2, \mathbb{C})$, rather than $SU(4)$ or $SL(4, \mathbb{R})$. The Dirac matrices are all unitary, however, and they generate a group of order $2^6 = 64$. A more natural analogue of Q_8 and G_{27}, however, is a so-called *extraspecial* group E_{128} of order 128 obtained by adjoining a complex conjugation operator, and thereby extending to 8 dimensions.

In principle, there is a choice as to whether the complex conjugation operator squares to $+1$ or -1, but it turns out that in order to fit the Dirac matrices together with the Gell-Mann matrices it is necessary to take the -1 case. Then the representation is quaternionic, which means it can also be written as a group of 4×4 quaternion matrices. The group E_{128} requires six generators, such as

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad i\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(8)

$$\begin{pmatrix} j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & j \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i\gamma^0\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

These generators are arranged in such a way that the three columns generate three mutually commuting copies of the quaternion group Q_8.

The names above are given according to the standard Bjorken-Drell convention [18], but many other arrangements are possible. For example, it is possible to choose $\gamma^1\gamma^2$, $\gamma^2\gamma^3$ and $\gamma^1\gamma^3$ as elements of one of the Q_8 factors, and $i\gamma_0$, $i\gamma_5$ and $\gamma_0\gamma_5$ as elements of the other Q_8 factor. This choice gives a useful mathematical separation of the properties that depend on a direction in space (such as spin and momentum) from those that do not (such as energy, mass and weak isospin).

It is well-known that the outer automorphism group of this group of order 2^7 is isomorphic to the Weyl group of type E_6, so that it is straightforward to calculate any necessary properties. In fact there are 120 subgroups isomorphic to Q_8, and they are all equivalent under the automorphism group, so we can take any one of them to represent the non-relativistic spin group as above. We then need to choose γ^0 and γ^5 from the nine elements of order 2 (up to sign) that commute with this copy of Q_8. The automorphism group acts transitively on these 9 elements, and the stabiliser of one of them acts transitively on the four that anti-commute with it. Hence all choices of Dirac matrices are mathematically equivalent. In particular, there is no reason to choose the Bjorken–Drell convention, or the 'chiral' convention (in which γ^0 and γ^5 are swapped, relative to the Bjorken–Drell convention), if other conventions provide simpler notation for physically important concepts. Our conventions, and the reasons behind them, will be explained in Section 3.1.

2. Extensions

2.1. **The binary tetrahedral group.** The quaternion group has an automorphism of order 3, that cycles the elements i, j and k, and that can be represented explicitly in the quaternion algebra by the matrix

$$\begin{aligned}(-1+I+J+K)/2 &= \frac{1}{2}\begin{pmatrix}-1+i & 1+i\\ -1+i & -1-i\end{pmatrix}\\ &= \frac{1+i}{2}\begin{pmatrix}i & 1\\ i & -1\end{pmatrix}\end{aligned} \tag{9}$$

This matrix extends the quaternion group Q_8 to a group of order 24 that is known as the binary tetrahedral group. The above matrices give a representation of this group inside $SU(2)$, but it also has a representation inside $U(2)$ that is not in $SU(2)$. These facts may be significant for the mixing of electrodynamics with the weak interaction.

At the more abstract group theory level, we have constructed a semi-direct product of Q_8 by Z_3, which we denote $Q_8 \rtimes Z_3$. The subgroup Q_8 generated by the matrices K, J, I in (4) is normal, but the subgroup Z_3 generated by the above matrix W is not. In place of the relations $IW = WI$, $JW = WJ$ that hold in the direct product, we have the relations $IW = WJ$ and $JW = WK$. The continuous analogue of Q_8 is $SU(2)$, and the continuous analogue of Z_3 is $U(1)$, so that the effect of this semi-direct product is to create a more complicated relationship between $SU(2)$ and $U(1)$ than the direct product $SU(2) \times U(1)$. In physics language this relationship is a 'mixing' of $SU(2)$ with $U(1)$ at the quantum level.

2.2. **A ternary representation.** In fact, the binary tetrahedral group can itself act as automorphisms of the group G_{27} of order 27 described above. We may first take generators for the quaternion subgroup Q_8 as

$$\frac{v-w}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & v & w\\ 1 & w & v\end{pmatrix}, \quad \frac{v-w}{3}\begin{pmatrix}1 & v & v\\ w & v & w\\ w & w & v\end{pmatrix}, \tag{10}$$

and check all the required relations explicitly. The calculations are all minor variants of the equation

$$\begin{aligned}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}1&1&1\\1&v&w\\1&w&v\end{pmatrix}&=\begin{pmatrix}1&v&w\\1&w&v\\1&1&1\end{pmatrix}\\ &=\begin{pmatrix}1&1&1\\1&v&w\\1&w&v\end{pmatrix}\begin{pmatrix}1&0&0\\0&v&0\\0&0&w\end{pmatrix}\end{aligned}\tag{11}$$

which shows how the first of the given automorphisms converts permutation matrices into diagonal matrices.

At this stage we have a group of order 216 that combines finite analogues of $SU(3)$ and $SU(2)$ into a single group. This is a semi-direct product $G_{27} \rtimes Q_8$, in which the subgroup G_{27} is normal, but the subgroup Q_8 is not. Since it is not a direct product, it does not give rise to a direct product $SU(3) \times SU(2)$, but instead 'mixes' the two gauge groups.

Finally we can add in a matrix with determinant v, such as

$$\begin{pmatrix}v&0&0\\0&1&0\\0&0&1\end{pmatrix}\tag{12}$$

in order to incorporate $U(1)$ as well, extending the quaternion group to the binary tetrahedral group. This gives us altogether a group of order $2^3.3^4 = 648$ (the triple cover of the Hessian group) that can be interpreted as a finite analogue of the complete gauge group $U(1) \times SU(2) \times SU(3)$ of the Standard Model.

But the finite group is not a direct product of three factors in the way that the Lie group must be. It is an iterated semi-direct product $Q_{27} \rtimes Q_8 \rtimes Z_3$. Thus the finite analogues of the three factors are mixed together in quite a complicated way, which may have important consequences for the 'mixing' of the different forces in the Standard Model. We have already discussed electroweak mixing in the context of $Q_8 \rtimes Z_3$, but here we have a much more complicated weak-strong mixing in $G_{27} \rtimes Q_8$, as well as electro-strong mixing in $G_{27} \rtimes Z_3$. Further details will be discussed below.

2.3. **A complex reflection group.** The group of order 648 is generated by the conjugates of (12), which has one eigenvalue v and two eigenvalues 1. An invertible matrix with a single non-trivial eigenvalue is known as a *reflection*. Real reflections have real eigenvalues, so the non-trivial eigenvalue is -1, but complex reflections can have any root of unity as an eigenvalue. Our group is therefore a 3-dimensional complex reflection group. There are 24 reflections altogether, in 12 'mirrors' defined by the complex vectors

$$\begin{array}{cccc}(t,0,0) & (1,1,1) & (v,1,1) & (w,1,1)\\ (0,t,0) & (1,v,w) & (1,1,v) & (1,w,1)\\ (0,0,t) & (1,v,w) & (1,v,1) & 1,1,w)\end{array}\tag{13}$$

where $t = v - w$. The two reflections in the mirror r are

$$x \mapsto x - (1-\lambda)\frac{x.r}{r.r}r\tag{14}$$

where $\lambda = v$ or $\lambda = w$.

The four mirrors given in the top row generate one of the nine copies of the binary tetrahedral group, acting on the 2-dimensional subspace perpendicular to $(0, 1, -1)$. In particular, this group is a 2-dimensional complex reflection group. The images of the (anti-Hermitian) Pauli matrices can be written as products of two reflections of order 3 (with opposite determinants). This representation is, however, quite different from the usual representation of the (Hermitian) Pauli matrices as real reflections in a complex 2-space, which does not appear in this model.

If we expand the complex 1-spaces to real 2-spaces, then the six complex scalar multiples of 12 complex 3-vectors become two real scalar multiples of 36 real 6-vectors, and the corresponding 36 real reflections generate the Weyl group of E_6. In other words, the group $G_{27} \rtimes Q_8 \rtimes Z_3$ is a subgroup of the Weyl group of E_6, and is in fact a maximal subgroup of the rotation part of the Weyl group. It is possible that this fact may have led to the appearance of E_6 symmetry in certain Grand Unified Theories (GUTs) such as [19, 20].

2.4. **A quaternionic reflection group.** The action of G_{27} on E_{128} is more difficult to calculate, but much of the work has been done for us in [15, 16], It is probably easiest to start from the quaternionic reflection group called S_3 in [15], which happens to be an extension of E_{128} by the rotation subgroup of the Weyl group of type E_6. Quaternionic reflections are given by essentially the same formula (14) as for complex reflections, with appropriate choices for the quaternionic scalar λ, except that in order to ensure that reflections are linear maps it is necessary to re-order the product as follows.

$$x \mapsto x - \frac{x.r}{r.r}(1-\lambda)r \tag{15}$$

This then ensures that replacing x by a scalar multiple μx gives the same scalar multiple on the right hand side. Moreover, replacing r by a scalar multiple μr replaces $x.r$ by $(x.r)\bar{\mu}$, so changes the scalar $1-\lambda$ to $1-\mu^{-1}\lambda\mu$. Hence reflections of order 2, with $\lambda = -1$, are invariant under this operation, but other reflections are not.

The group G_{27} can be generated by the products of pairs of reflections of order 2 in the mirrors defined by

$$(2,0,0,0), (\pm 1,1,1,1), (\pm 1,i,j,k), (\pm 1,j,k,i), (\pm 1,k,i,j) \tag{16}$$

Indeed, we can take the products of the first reflection with the other 8 to represent the 8 Gell-Mann matrices. Generators for the full group of order 648 can then be taken as

$$\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & k \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{17}$$

The first matrix above represents one of the Gell-Mann matrices in G_{27}, the second one of the Pauli matrices in Q_8, and the last is a generator for Z_3.

There is some ambiguity about which matrix to use for Z_3, since it can be mixed with the scalar part of G_{27}, also of order 3. In the Standard Model, this mixing is done with two copies of $U(1)$, and is therefore in principle represented by an arbitrary angle, called the CP-violating phase of the Cabibbo–Kobayashi–Maskawa (CKM) matrix [21, 22]. The discrete model at this stage does not tell us anything about the value of this angle.

3. The structure of the combined group

3.1. **Dirac matrices.** The Dirac matrices for three generations generate the extraspecial group E_{128}, which is a commuting product of three copies of Q_8. There are 40 different ways in which the group can be written as such a product, but all are equivalent under the action of the Weyl group of E_6 as the outer automorphism group. Hence we may take the factorisation as the columns in (8) as the 'standard' one. However, the appropriate interpretations of the three factors are far from clear.

For example, as a first tentative suggestion, we can separate the quaternions i, j, k to generate one copy of Q_8 to describe the three generations, and use the real matrices for two other copies of Q_8 to describe spin and weak isospin. It therefore makes sense to separate the spin copy of Q_8 from the weak isospin copy, as follows:

$$\gamma^1\gamma^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \gamma^0\gamma^5 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad i\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

This allocation shows that we can only get an energy term in γ^0 if we attach a generation label i, j or k. Indeed, the same is true for the momentum terms in γ^1, γ^2 and γ^3.

To see this, we can take generators for the real Clifford algebra $Cl(3,1)$ as follows:

$$i\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$
$$i\gamma^1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad i\gamma^0 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{19}$$

and extend to the complex Clifford algebra, or $Cl(4,1)$, by adjoining

$$\gamma^5 = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \tag{20}$$

With our conventions, the matrices γ^0, γ^1, γ^2, γ^3 and γ^5 are all real matrices times i. This enables us to reduce from the proposed three-generation model of Dirac matrices to the standard one-generation model by choosing an appropriate imaginary 'scalar' i. But of course, the symmetry allows us to choose j or k for the other two generations, if this our choice of interpretation. In any case, the choice of real matrices for a model without a generation label (or perhaps better, a model without mass) requires the reversed signature, but only $Cl(1,3)$.

The full 4×4 quaternion matrix algebra can also be interpreted as a Clifford algebra for a 6-dimensional real space with signature $(6, 0)$, $(5, 1)$, $(2, 4)$ or $(1, 5)$. Many models of this type have been proposed [23, 24, 25, 26], including the Pati–Salam model based on $SU(4) = Spin(6)$, and the twistor theory of Penrose, based on the Clifford algebra $Cl(2, 4)$ and the corresponding spin group isomorphic to $SU(2, 2)$. Twistors can equally well be given the opposite signature, so that the matrix algebra underlying $Cl(4, 2)$ is the algebra of 8×8 real matrices. Models of this type have also been proposed [27], but it would appear from this discussion that the signature $Cl(5, 1)$ is most likely to provide the closest fit to the Standard Model, and to observed physics, depending of course on the details of the physical interpretation of the mathematical structure.

The Dirac matrices above now act on columns of four quaternions, as a generalisation to three generations of the standard Dirac spinors, that are columns of four complex numbers. The projections with $1 \pm \gamma^5$ onto left-handed and right-handed (Weyl) spinors are a little more complicated than in the standard conventions, and also depend on the generation. But this is a small price to pay for incorporating the generation structure of the elementary fermions directly into the Dirac spinors. At the same time, we see that three generations only require twice as many spinors as one generation, as has also been pointed out in [17, 28]. This refutes one of the arguments in [29] that purports to prove the non-existence of an E_8 model of fundamental physics.

3.2. **Gell-Mann matrices.** The full set of 8 Gell-Mann matrices can be taken as the following four matrices and their conjugate transposes:

$$\begin{array}{c} \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & i & j & k \\ i & 1 & k & -j \\ j & -k & 1 & i \\ k & j & -i & 1 \end{pmatrix}, \\ \frac{1}{2}\begin{pmatrix} -1 & j & k & i \\ j & 1 & i & -k \\ k & -i & 1 & j \\ i & k & -j & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} -1 & k & i & j \\ k & 1 & j & -i \\ i & -j & 1 & k \\ j & i & -k & 1 \end{pmatrix}. \end{array} \tag{21}$$

The two real matrices here correspond to the two colourless gluons, and the other six to the coloured gluons. The group of order 27 generated by these matrices contains a centre of order 3, generated by

$$\begin{pmatrix} v & 0 & 0 & 0 \\ 0 & 0 & v & 0 \\ 0 & 0 & 0 & v \\ 0 & v & 0 & 0 \end{pmatrix} \tag{22}$$

where $v = (-1 + i + j + k)/2$. This matrix acts on the Dirac matrices to permute the three generations defined by i, j and k. At the same time, it permutes the three directions of spin, defined by $\gamma^1\gamma^2$, $\gamma^2\gamma^3$ and $\gamma^3\gamma^1$.

But it is important to note that replacing v by w in (22) can only be done if the coordinate permutation is also inverted. The chirality of the weak interaction ultimately arises from this important fact, as we shall see when we consider the representation of the Pauli matrices.

3.3. **Pauli matrices.** The (anti-Hermitian) Pauli matrices generating Q_8 are represented by the diagonal matrices

$$\begin{pmatrix}1&0&0&0\\0&i&0&0\\0&0&j&0\\0&0&0&k\end{pmatrix}, \quad \begin{pmatrix}1&0&0&0\\0&j&0&0\\0&0&k&0\\0&0&0&i\end{pmatrix}, \quad \begin{pmatrix}1&0&0&0\\0&k&0&0\\0&0&i&0\\0&0&0&j\end{pmatrix}, \tag{23}$$

acting on the spinors. These matrices commute with (22), but are permuted by both the 'scalar' v and the coordinate permutation separately. The scalar acts by permuting the three generations, but leaving the spin direction alone, while the permutation acts on the spin direction but not on the generation. The fact that the matrix (22) couples the cyclic ordering of the three generations to the cyclic ordering of the three directions of spin is what is known as the *chirality* of the weak interaction. Conventionally this chirality is *left-handed*, so that the copy of $SU(2)$ generated by the Pauli matrices above is denoted $SU(2)_L$.

The way chirality is implemented in the Standard Model is by restricting to the third component of weak isospin, represented say by the first matrix in (23), and realising j as complex conjugation, so that the bottom half of the Dirac spinor behaves as a Weyl spinor that is the complex conjugate of the top Weyl spinor. Thus the proposed model is a straightforward generalisation of the Standard Model at this point, and incorporates all three components of weak isospin.

We can now extend from Q_8 to the binary tetrahedral group by adjoining any one of the three matrices

$$\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{pmatrix}, \quad \begin{pmatrix}v&0&0&0\\0&v&0&0\\0&0&v&0\\0&0&0&v\end{pmatrix}, \quad \begin{pmatrix}v&0&0&0\\0&0&0&v\\0&v&0&0\\0&0&v&0\end{pmatrix}. \tag{24}$$

As noted above, the first of these acts on the colours, and on the direction of spin, but not on the generations. The second one acts on the colours and the generations, but not on the direction of spin. The third one acts on all three.

4. Representations and characters

4.1. **Continuous gauge groups.** So far we have only looked at the combinatorial structure of the group, which gives a qualitative but not quantitative picture of the underlying physics. In order to introduce measurable physical quantities such as mass, momentum and energy, and allocate them to elementary (or composite) particles we need to introduce real (or complex or quaternionic) representations, and use real (or complex or quaternionic) gauge groups to choose coordinates for the representations. There are four fundamental representations, described above, given by the embeddings of finite groups in Lie groups as follows:

$$\begin{aligned} Z_3 &\subset U(1)\\ Q_8 \rtimes Z_3 &\subset Sp(1) \cong SU(2)\\ G_{27} \rtimes Q_8 \rtimes Z_3 &\subset U(3)\\ E_{128} \rtimes G_{27} \rtimes Q_8 \rtimes Z_3 &\subset Sp(4) \end{aligned} \tag{25}$$

The first three are closely related to the gauge groups $U(1)$, $SU(2)$ and $SU(3)$ of the Standard Model, while the last is not in the Standard Model at all.

In each case, the real scalar factor has been taken out of the gauge group, since it only defines a unit of measurement, rather than a genuine physical property. In the case of $U(1)$ acting on a complex 1-dimensional representation, we are left therefore with one physical parameter to measure, which can be represented as an angle, or phase, in the complex plane. In the case of $SU(2)$ acting on a 1-dimensional quaternionic representation, there are three such angles. Therefore by studying the representations of the binary tetrahedral group $Q_8 \rtimes Z_3$, we might expect to find four of the fundamental mixing angles of the Standard Model, including the electro-weak mixing angle (Weinberg angle). A tentative identification of these four angles is obtained in [9].

In the case of $U(3)$, there are five independent angles associated with the 3-dimensional complex representation, bringing the total number of mixing angles up to 9, which accounts for the Weinberg angle plus four each in the CKM and PMNS (Pontecorvo–Maki–Nakagawa–Sakata) matrices [30, 31]. We have a further 15 parameters in the coordinates of the 4-dimensional quaternionic representation on the spinors. If we take at face value the proposed use of this representation for a theory of gravity, then we should expect these parameters to be masses of elementary particles. The Standard Model nowadays has 15 mass parameters, including 12 for the elementary fermions, plus the W, Z and Higgs bosons.

Finally, note that by restricting the 4-dimensional representation to *subgroups* Z_3, $Q_8 \rtimes Z_3$ and $G_{27} \rtimes Q_8 \rtimes Z_3$ we obtain the first three representations again. It is therefore entirely possible that the 9 mixing angles of the Standard Model are redundant parameters, and can be derived from the 15 masses. This is known to be the case for the electro-weak mixing angle, which is derived from the mass ratio of the W and Z bosons. In [9] three more such derivations are proposed.

4.2. **Clifford theory.** The standard method of constructing representations and characters of semi-direct products is Clifford theory [32], which is also used more generally whenever a group has a normal subgroup. Since the group $E_{128} \rtimes G_{27} \rtimes Q_8 \rtimes Z_3$ has 8 normal subgroups, forming a single chain, we can work our way down the chain, from the largest normal subgroup to the smallest, adding new representations at each stage. The method is standard, and the calculations straightforward, so I do not give many details.

At the first stage, we have only the trivial quotient group, and the trivial representation. At the second stage, we have a quotient Z_3, which adds a 2-dimensional real representation, or equivalently a 1-dimensional complex representation and its complex conjugate. At the third stage, the quotient group is the alternating group A_4 on four letters, and the new representation is the monomial representation of the group acting by conjugation on the three Pauli matrices, with signs attached.

At the fourth stage, we have the fundamental pseudoreal representation as quaternions, with Z_3 as the corresponding 'inertial quotient', so that the other representations at this stage are obtained by tensoring with the representations of Z_3. This gives us a pair of complex conjugate two-dimensional representations, or equivalently a real four-dimensional representation. Putting all these stages together gives us the representations of the binary tetrahedral group, which are very well known and have been widely studied. The character table can be found in [33, p.404] and is reproduced here for convenience, in a shorthand form in which conjugacy classes that are scalar multiples of each other are grouped together:

(26)

	± 1	K	$\pm W$	$\pm W^2$
$\mathbf{1}a$	1	1	1	1
$\mathbf{1}b$	1	1	v	w
$\mathbf{1}c$	1	1	w	v
$\mathbf{3}$	3	-1	0	0
$\mathbf{2}a$	± 2	0	∓ 1	∓ 1
$\mathbf{2}b$	± 2	0	$\mp v$	$\mp w$
$\mathbf{2}c$	± 2	0	$\mp w$	$\mp v$

Of particular significance here is that the three faithful representations are obtained by taking the fundamental representation $\mathbf{2}a$ and tensoring with the representations $\mathbf{1}a$, $\mathbf{1}b$ and $\mathbf{1}c$ of Z_3. This type of pattern is a general feature of Clifford theory, and is a generalisation of the decomposition of representations of direct products of groups as tensor products of representations of the factors. In particular, we have a discrete version of the decomposition of representations of $U(1) \times SU(2)$, but with the 'mixing' of $U(1)$ with $SU(2)$ already incorporated into the mathematical structure. A combinatorial version of the mixing occurs already at the group-theoretical level, but a quantitative estimate of the mixing angle can only be obtained by looking in detail at the coordinates of the representations.

4.3. **The Hessian group.** Coming now to G_{27}, whose character table can be found in [33, p.400] we first take a 1-dimensional complex representation, with inertial quotient Z_3, and induce up to the whole group $G_{27} \rtimes Q_8 \rtimes Z_3$, to get an 8-dimensional real representation, together with its tensor products with the representations of Z_3. The scalars in G_{27} act trivially on these representations, so they are representations of a group $(Z_3 \times Z_3) \rtimes Q_8 \rtimes Z_3$ of order 216, which has been known as the Hessian group since the 1870s.

The table of characters of the complex irreducible representations of the Hessian group is as follows. The vertical lines delineate normal subgroups, so that the first two columns comprise the normal subgroup $G_{27}/Z_3 \cong Z_3 \times Z_3$, generated by the modified Gell-Mann matrices modulo scalars, and the next two columns comprise the quaternion group Q_8 generated by the Pauli matrices. The horizontal lines separate cohorts of characters for the various quotient groups. The top two rows contain the centralizer orders and the sizes of the conjugacy classes, respectively.

(27)

216	27	24	4	18	9	6	18	9	6
1	8	9	54	12	24	36	12	24	36
1	1	1	1	1	1	1	1	1	1
1	1	1	1	v	v	v	w	w	w
1	1	1	1	w	w	w	v	v	v
3	3	3	-1	0	0	0	0	0	0
2	2	-2	0	-1	-1	1	-1	-1	1
2	2	-2	0	$-v$	$-v$	v	$-w$	$-w$	w
2	2	-2	0	$-w$	$-w$	w	$-v$	$-v$	v
8	-1	0	0	2	-1	0	2	-1	0
8	-1	0	0	$2v$	$-v$	0	$2w$	$-w$	0
8	-1	0	0	$2w$	$-w$	0	$2v$	$-v$	0

For some purposes it is useful to replace a pair of complex conjugate characters by their real sum, and to fuse a conjugacy class with its inverse class. This give us a simplified table

$$
\begin{array}{c|cc|cc|ccc|}
 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
U(1)=SO(2) & 2 & 2 & 2 & 2 & -1 & -1 & -1 \\ \hline
SO(3) & 3 & 3 & 3 & -1 & 0 & 0 & 0 \\ \hline
SU(2) & 2 & 2 & -2 & 0 & -1 & -1 & 1 \\
U(2) & 4 & 4 & -4 & 0 & 1 & 1 & -1 \\ \hline
SU(3) & 8 & -1 & 0 & 0 & 2 & -1 & 0 \\
U(3) & 16 & -2 & 0 & 0 & -2 & 1 & 0
\end{array}
\tag{28}
$$

The first column gives the name of the compact Lie group generated by the finite group, from which we can see a close relationship to the gauge group of the Standard Model. The even-numbered rows contain the basic building blocks $U(1)$, $SU(2)$ and $SU(3)$ respectively, and the other rows contribute to the 'mixing' between these three groups.

4.4. **The faithful characters.** Finally we take a 3-dimensional complex faithful representation of G_{27}, with inertial quotient $Q_8 \rtimes Z_3$, so that Clifford theory implies that we must now tensor with all the representations of the latter group.

Let us arrange the conjugacy classes grouped together into scalar multiples. The we only need to specify the character value on one of these classes, and multiply by the appropriate scalars to get the others. In three cases an element is conjugate to its scalar multiples, so that the character values are all 0, while in the other seven they are not. Writing t for $v - w = \sqrt{-3}$, the 14 faithful complex characters are the following together with their complex conjugates.

$$
\begin{array}{|cc|cc|cccccc|}
3 & 0 & -1 & -1 & t & 0 & t & -t & 0 & -t \\
3 & 0 & -1 & -1 & vt & 0 & vt & -wt & 0 & -wt \\
3 & 0 & -1 & -1 & wt & 0 & wt & -vt & 0 & -vt \\ \hline
9 & 0 & -3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
6 & 0 & 2 & 0 & -t & 0 & t & t & 0 & -t \\
6 & 0 & 2 & 0 & -vt & 0 & vt & wt & 0 & -wt \\
6 & 0 & 2 & 0 & -wt & 0 & wt & vt & 0 & -vt
\end{array}
\tag{29}
$$

It is not really necessary to write out this table in full, since, as noted above, the seven rows are obtained by multiplying the first row with the seven irreducible characters of the binary tetrahedral group, that is the first seven characters of (27). If we take the latter to describe properties of leptons, then (29) describes the corresponding properties of quarks.

5. Action on spinors and Dirac matrices

5.1. **Factorisation of the Dirac matrices.** As noted in Section 1.4, the quaternionic Dirac matrices factorise in 40 different ways as a commuting product of three copies of Q_8. I have tentatively proposed to identify one copy as a generation or flavour group, and another as a (non-relativistic) spin group. The third therefore has to deal with both relativistic corrections and weak isospin. This interpretation may or may not be suitable: it all depends on how the Gell-Mann and Pauli matrices act on the Dirac matrices in this model.

First note that the Gell-Mann matrices preserve this decomposition, but the Pauli matrices do not. This means that the Pauli matrices (representing the weak interaction) mix the flavour group with the spin group, in particular. This is the characteristic experimental property of beta decay, as measured by the Wu experiment [34] in 1957: the first-generation electron/neutrino flavour is coupled to the direction of spin and the direction of momentum. Hence this mathematical property of the model is fundamentally linked to the 'chirality' of the weak interaction.

To illustrate this in more detail, consider the first of the Pauli matrices represented in (23), acting by conjugation on the quaternionic Dirac matrices in (8), but with the labelling of the real part given in (18) and (19). The first row of (8) then maps to

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & j & 0 \\ 0 & 0 & 0 & j \\ j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -j & 0 \\ 0 & 0 & 0 & j \\ -j & 0 & 0 & 0 \\ 0 & j & 0 & 0 \end{pmatrix}. \tag{30}$$

In particular, the 'flavour' matrix has been multiplied by the diagonal matrix $\gamma^0\gamma^2$, which combines elements from both the 'spin' and 'weak isospin' copies of Q_8, as we would expect from experiment. The 'spin' matrix has acquired a factor of j, corresponding to complex conjugation in the Standard Model, that swaps the left-handed and right-handed Weyl spinors. The 'weak isospin' matrix has also acquired a factor of j, but how this should be interpreted I am not sure.

5.2. **Action of Gell-Mann matrices.** We now consider the Gell-Mann matrices (21). The first pair, that is the real matrices, representing colourless gluons, commute with the 'flavour' Q_8, consisting of 'scalar' matrices. Since they have order 3, they cannot swap the other two factors, so they fix both the 'spin' and 'weak isospin' copies of Q_8. The precise action on spin can be calculated as follows:

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{31}$$

which shows that these gluons have the effect of changing the direction of spin from the z direction to the x and y directions.

The action on the third copy of Q_8 is similar:

$$\begin{aligned} &\frac{1}{2}\begin{pmatrix} -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned} \tag{32}$$

Here we see a conversion from $\gamma^0\gamma^5$ to $i\gamma^5$ (and also to $i\gamma^0$), which indicates a mixing of some kind with the weak interaction, as well as a conversion of (kinetic) energy into some other form. This may perhaps describe the way in which the strong force creates mass for pseudoscalar mesons.

5.3. **Colour.** The action of the coloured gluons is a good deal more complicated, since they act by permuting the three copies of Q_8. Thus they map spin symmetries to flavour symmetries and weak isospin symmetries, which seems to imply that the entire structure of spacetime breaks down once we look at the internal structure of a baryon. In other words, at distances less than about 10^{-15}m, it no longer makes sense to talk about 'space' at all. Dividing by the speed of light to convert distance to time, we obtain a cutoff of around 3×10^{-24}s below which it does not make sense to talk about 'time'. This is roughly (to within a factor of 2) the lifetime of the most unstable mesons and baryons in the Standard Model. Note however that the lifetime of the W and Z bosons is an order of magnitude smaller, indicating that the process of beta decay of a neutron takes place on this even smaller scale of both time and space, where spacetime may behave strangely.

It is straightforward to compute the action of the coloured gluons on the Dirac matrices, and verify that the action is compatible with (7). Hence the latter notation provides a simpler way to see the action, if we interpret the three complex coordinates as representing the three Q_8 factors. In this way the diagonal matrices represent the colourless gluons, and the coloured gluons permute the three factors.

5.4. **Action on spinors.** The above analysis shows that the finite group $G_{27} \rtimes Q_8 \rtimes Z_3$ takes the 'spin' copy of Q_8 to a total of 12 distinct copies within the quaternionic Dirac matrices. Moreover, in each case there is still a choice of 9 elements to represent the energy in γ^0. In other words, in this model the nuclear forces act to change the shape of spacetime, at least locally. In principle this gives us a mechanism for relating the mass that is created or destroyed by the weak and strong forces (and potentially also the Higgs field) to the shape of spacetime that determines the gravitational field in General Relativity. However, an analysis of the Dirac matrices from this point of view will be enormously complicated, so I make no attempt to do this here.

Indeed, before considering such action on the quaternionic Dirac *matrices* we should investigate in much more detail the action on the quaternionic Dirac *spinors*, which is of course quite distinct. Whilst the Standard Model splits the complex Dirac spinor into two pieces, the left-handed and right-handed Weyl spinors, the splitting here is into four pieces, which do not naturally have a complex (Weyl spinor) structure, but rather a quaternionic structure. We can recover the standard description by choosing a complex structure, which breaks the symmetry between the three generations, and at the same time defines a particular copy of the Lorentz group among the many that are available.

The hope is, therefore, that by not breaking the symmetry of the quaternions, we can also avoid the necessity to choose a definition of inertial frame. Or, to put it another way, the finite group implies that there is a 'mixing' between the generation symmetry and the choice of inertial frame. This type of mixing between particle physics and gravity was first considered by Einstein [41] in 1919, but has not found favour in mainstream physics, in which the separation between gravity and particle physics has been rigorously maintained.

Nevertheless, the choice of the laboratory frame of reference as 'inertial' is fraught with problems, particularly in cases where an experiment spans two different laboratories, with very different definitions of inertial frame. The model proposed here implies that such a change in inertial frame should be reflected in a change in 'generation' of fermions. Of course, there is no experimental evidence that the generation of an electron is affected in this way, but there is a lot of experimental evidence that the generation (flavour) of a neutrino is [42]. Indeed, it is shown in [9] that by inserting the masses of the three generations of electron into the representations of the Z_3 quotient of our finite group, the experimental value of the mixing angle between electron neutrino and muon neutrino can be obtained. In other words, the model implies that the electron and neutrino generations are coupled together in a more complicated way than in the Standard Model.

As regards quarks, the evidence is more equivocal, and would require more detailed analysis. But again, the model we are considering here allows us to put the masses of three generations of electron into a quaternion representation of $Q_8 \rtimes Z_3$, together with a proton or neutron, from which the correct experimental value for the mixing angle between second and third generation quarks can be calculated [9]. Again, this suggests a more complicated relationship between lepton and quark generations than is implied by the Standard Model.

6. Conclusion

The suggestion that physics might be fundamentally discrete, and that the apparently continuous behaviour we often observe is an illusion, goes back a long way. In the context of quantum mechanics, this suggestion was made by Einstein [35] as early as 1935, but no discrete theory was found to rival the continuous models based on the Schrödinger equation and then the Dirac equation. The Standard Model of Particle Physics, largely completed by the mid 1970s, is entirely based on continuous groups, so much so that the emergence of discrete experimental observations is still not explained in a totally convincing way. Theories beyond the Standard Model are almost without exception continuous theories.

A few lone voices, such as 't Hooft [36], still speak out in favour of Einstein's dream of a discrete foundation, even if this has to be at the Planck scale. But a convincing discrete model is still lacking, and few people are seriously looking for one. The ones I have put forward in [9, 37, 38, 39, 10, 40] all suffer from drawbacks of various kinds, most notably the lack of any convincing analogue of the colour $SU(3)$, which is an essential and very successful part of the Standard Model. Nevertheless, it is a mathematical fact that finite groups are *much* more complicated than Lie groups, and quite small finite groups can model properties that cannot be modelled with Lie groups at all.

In this paper, therefore, I have tried to remedy this defect by explicitly proposing a finite analogue or precursor of $SU(3)$. At a combinatorial (qualitative) level it has many desirable properties that mirror known experimental properties of elementary particles, in particular the chirality of the weak interaction, and the mixing of electromagnetism and the weak and strong nuclear forces. Quantitative analysis of masses, mixing angles and probability amplitudes is beyond the scope of this paper, as it requires detailed investigations of coordinates for the representations.

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