ON THE EMBEDDING OF C_3 IN E_8

ROBERT A. WILSON

Abstract. I investigate the structure of E_8 under the action of the subalgebra/subgroup $A_1 + G_2 + C_3$, as a potential route to unification of the fundamental forces of nature into a single algebraic structure. The particular real form $E_{8(-24)}$ supports a decomposition into compact G_2 plus split $A_1 + C_3$, which allows a restriction from G_2 to $SU(3)$ for QCD, together with split $SL_2(\mathbb{R})$ to break the symmetry of the weak interaction and give mass to the bosons. The factor C_3 contains a copy of the Lorentz group $SL_2(\mathbb{C})$ and extends the 'spacetime' symmetries to the full group of symplectic symmetries of $3 + 3$-dimensional phase space.

1. Introduction

A number of E_8 models of fundamental physics have been proposed in recent years [1, 2, 3, 4, 5], but none of them has been sufficiently compelling to persuade large numbers of people that they are useful. The key issue is how to split up the symmetries of E_8 to get something that looks like the Standard Model, and in particular, how to do this in a reasonably 'natural' way. In addition there are a number of technical issues which cause a lot of trouble, particularly to do with complex structures and chirality, and with implementing three generations of fermions when there appears on the face of it to be only enough room for two [6].

The approach to 'naturality' taken in [1] is to take the Freudenthal–Tits 'magic square' [7, 8, 9] as a guide. The Lie structure of the magic square is

$$\begin{array}{cccc} A_1 & A_2 & C_3 & F_4 \\ A_2 & A_2 + A_2 & A_5 & E_6 \\ C_3 & A_5 & D_6 & E_7 \\ F_4 & E_6 & E_6 & E_8 \end{array} \tag{1}$$

and most emphasis has been put on the fourth row and the fourth column, where the exceptional Lie groups are found. However, the other entries in the table are also interesting. For example, the route taken in [1] from top left to bottom right goes via A_2, A_5 and E_6 in the second row (see also [10, 11]).

The corresponding 'magic square' for 2×2 matrices is

$$\begin{array}{cccc} B_0 & B_1 & B_2 & B_4 \\ B_1 & D_2 & D_3 & D_5 \\ B_2 & D_3 & D_4 & D_6 \\ B_4 & D_5 & D_6 & D_8 \end{array} \tag{2}$$

where I use the notation B and D to emphasise that these are all spin groups, with $B_n = Spin(2n + 1)$ and $D_n = Spin(2n)$.

Date: First draft: 20th April 2024; this version; 22nd April 2024.

However, I would like to draw particular attention to the isomorphisms between spin groups (types B and D) and unitary (type A) and symplectic (type C) groups:

$$\begin{aligned} B_1 &= A_1, \\ D_2 &= A_1 + A_1, \\ B_2 &= C_2, \\ D_3 &= A_3. \end{aligned} \tag{3}$$

The isomorphism $B_1 = A_1$ gives the basic fact that $Spin(3) \cong SU(2)$, which is the foundation of quantum mechanics (QM), while $D_2 = A_1 + A_1$ extends this to $Spin(3,1) \cong SL(2,\mathbb{C})$ for relativistic QM. Also, $D_3 = A_3$ plays a significant role in Grand Unified Theories (GUTs) from 1974 onwards [12, 13, 14], in various real forms including $Spin(6) \cong SU(4)$, $Spin(4,2) \cong SU(2,2)$ and $Spin(3,3) \cong SL_4(\mathbb{R})$. On the other hand, the isomorphism $B_2 = C_2$ has been relatively neglected, although it lies at the heart of the AdS/CFT correspondence [15].

This neglect is strange, because the isomorphism $D_2 = A_1 + A_1$ is insufficient to explain why the Dirac algebra is a complex rather than real Clifford algebra, whereas the isomorphism $B_2 = C_2$ has enough room to include γ_5, as is required for electro-weak unification. Extending then to the 3×3 case, we extend C_2 to C_3, which looks a lot more 'natural' than extending B_2 to C_3. It is worth noting also that all of the exceptional isomorphisms listed here arise from the triality automorphism of D_4, which links together all the groups in the table that are not in the fourth row or fourth column. In this note, therefore, I concentrate on the third row of the magic square [16, 17], and especially the first group, $Sp_6(\mathbb{R})$, that has a classical interpretation as the symmetry group of phase space [18].

2. Embedding C_2 in D_8

In the semi-split version of the magic square, both the compact and split real forms of C_2 occur, but because I want to use C_2 to implement the Dirac algebra, I want the split form, that is the group $Spin(3,2) \cong Sp_4(\mathbb{R})$. Embedding into $Spin(12,4)$ we see the centralizer $Spin(9,2)$, which we can split into three pieces, $Spin(6)$, $Spin(3)$ and $Spin(2)$, if we want to get the Standard Model gauge groups $SU(3)$, $SU(2)$ and $U(1)$. However, the reason for this splitting will not become clear until we consider the embedding of C_3 in E_8.

In the notation of [1, 19], we need to choose a copy of the split quaternions $\mathbb{H}'$ in the split octonions, say the copy with basis U, K, L, KL. Then the corresponding copy of $Spin(3,2)$ acts on the indices $u := 1, U, K, L, KL$, leaving IL, JL for $Spin(2)$, and i, j, k, il, jl, kl for $Spin(6)$ and/or $SU(3)$, so that $Spin(3)$ acts on l, I, J. In particular we see some symmetry-breaking for $Spin(3)$, generated by X_{lI}, X_{lJ} and $D_{I,J}$, already in the notation. Now for the Dirac part of the algebra, the labels u, U, K, L, KL correspond to the matrices $\gamma_1, \gamma_2, \gamma_3, \gamma_0$ and γ_5 respectively, and the products of pairs of these gamma matrices generate the algebra $\mathfrak{so}(3,2)$:

$$\begin{aligned} &X_1, X_K, D_K, \\ &X_L, D_L, D_{K,L}, \\ &X_{KL}, D_{KL}, D_{K,KL}, D_{L,KL}, \end{aligned} \tag{4}$$

in which the first row contains the rotations in $SL_2(\mathbb{C})$, and the second row contains the boosts, while the third row contains a Lorentzian 4-vector.

Since this algebra is quaternionic rather than octonionic, it can be written as ordinrary 2×2 anti-Hermitian quaternion matrices, which make it easier to understand the structure. The Xs are off-diagonal, the single-index Ds are diagonal traceless, and the double-index Ds add the imaginary traces:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}, \begin{pmatrix} K & 0 \\ 0 & -K \end{pmatrix},$$
$$\begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix}, \begin{pmatrix} L & 0 \\ 0 & -L \end{pmatrix}, \begin{pmatrix} KL & 0 \\ 0 & KL \end{pmatrix},$$
$$\begin{pmatrix} 0 & KL \\ KL & 0 \end{pmatrix}, \begin{pmatrix} KL & 0 \\ 0 & -KL \end{pmatrix}, \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \tag{5}$$

These matrices can be regarded as a split quaternionic version of the Pauli matrices, which occur in the first row, in the mathematicians' anti-Hermitian convention rather than the physicists' Hermitian convention. The last matrix in the third row is the scalar matrix that is the product of the three Pauli matrices (Hermitian convention).

3. Extending to C_3

To extend from C_2 to C_3 we have to generalise the X terms to Y and Z, and the D terms to E:

$$\begin{aligned} &Y_1, Y_K, Y_L, Y_{KL}, \\ &Z_1, Z_K, Z_L, Z_{KL}, \\ &E_K, E_L, E_{KL}. \end{aligned} \tag{6}$$

Equivalently, we extend from 2×2 matrices to 3×3, to obtain a split quaternionic version of the Gell-Mann matrices [20]. It turns out that the 'scalar' 2×2 matrices extend to traceless 3×3 matrices, in the same way that the Gell-Mann matrices are traceless. In the notation of [19], this condition is enforced by the identity:

$$D_p + E_p + F_p = 0 \tag{7}$$

for all single index p, here K, L and KL. I make no claims as to how these Gell-Mann matrices should be interpreted, or whether they have anything to do with the Gell-Mann matrices used in QCD [21].

In fact, the compact part of this group $Sp_6(\mathbb{R})$ is a copy of $U(3)$ generated by all the elements that have an even number of copies of L in their labels:

$$X_1, X_K, Y_1, Y_K, Z_1, Z_K, D_K, E_K, D_{L,KL} \tag{8}$$

where again the double-index Ds add an imaginary trace to the matrices. As matrices, these are just the anti-Hermitian matrices of the complex subalgebra $\mathbb{C}$ of $\mathbb{H}'$. There are also two copies of $GL_3(\mathbb{R})$ obtained by replacing K by L or KL (or indeed any linear combination of the two):

$$\begin{aligned} &X_1, X_L, Y_1, Y_L, Z_1, Z_L, D_L, E_L, D_{K,KL}; \\ &X_1, X_{KL}, Y_1, Y_{KL}, Z_1, Z_{KL}, D_{KL}, E_{KL}, D_{K,L}. \end{aligned} \tag{9}$$

These are anti-Hermitian matrices over the respective copies of the split complex numbers $\mathbb{C}'$ in $\mathbb{H}'$. Leaving off the double-index Ds restricts to $SL_3(\mathbb{R})$, and the subtle relationships between these two copies of $SL_3(\mathbb{R})$ and $SU(3)$ will play an important role in this paper.

4. The centralizer of C_3

The centralizer of $Sp_6(\mathbb{R})$ in E_8 is a group of type $A_1 + G_2$, in which the copy of G_2 is compact, acting on the indices i, j, k, l, il, jl, kl, and the copy of A_1 is split, acting on the indices I, J, IL, JL in one of its chiral spinor (weak isospin?) representations. This copy of A_1 is obviously not the same as the copy, acting on l, I, J, that we suggested in Section 2 simply by looking in D_8. However, we effectively chose a copy of $\mathfrak{su}(2) + \mathfrak{u}(1)$ acting on l, I, J, IL, JL, so that by 'mixing' $D_{I,J}$ with $D_{IL,JL}$ we obtain one of the elements of the centralizer. To get the rest we need to replace X_{lI} and X_{lJ} by one combination of $D_{I,IL}$ and $D_{J,JL}$ and another combination of $D_{I,JL}$ and $D_{J,IL}$.

In other words, the conversion between these two copies of A_1, one of which is compact and the other split, is very reminiscent of the 'symmetry-breaking' of the weak $SU(2)$ in the Standard Model, that converts from a 'primordial' massless $SU(2)$ to $SL_2(\mathbb{R})$ via the complexification $SL_2(\mathbb{C})$, in order to give the intermediate vector bosons non-zero masses. Thus the E_8 model provides a fundamental mathematical reason for this symmetry-breaking, namely enforcement of the condition that the gauge group must commute not only with D_2, as the Coleman–Mandula theorem [22] requires, or with C_2, as the embedding in D_8 requires, but with the whole of C_3.

To see the details of how these two copies of A_1 are related to each other, we can work in the group they generate, which is another copy of $Spin(3, 2)$, acting on the labels l, I, J, IL, JL, and commuting with the first copy. The ten dimensions of this group are represented by

$$\begin{aligned} &X_{lI}, X_{lJ}, D_{I,J}, D_{IL,JL}, \\ &X_{lIL}, X_{lJL}, D_{I,IL}, D_{I,JL}, D_{J,IL}, D_{J,JL}. \end{aligned} \tag{10}$$

Since we are genuinely using the octonions at this point, it is not possible to write these elements of the algebra as ordinary matrices. However, the notation of [19] may help to visualise what is going on. The generators of the two copies of A_1 are related as follows (where the signs are determined by the chirality, or equivalently by the embedding in G_2):

$$\begin{aligned} D_{I,J} &\to D_{I,J} - D_{IL,JL} \\ X_{lI} &\to D_{I,IL} - D_{J,JL} \\ X_{lJ} &\to D_{J,IL} + D_{I,JL}. \end{aligned} \tag{11}$$

Turning now to the remaining factor, which is presumably related to the strong force, we compare the group $Spin(6)$ that appears in the centralizer of $Spin(3, 2)$, with the group G_2 that appears in the centralizer of $Sp_6(\mathbb{R})$. After replacing the original (unbroken symmetry) copy of A_1 by the chiral copy on I, J, IL, JL, we no longer require the label l for the weak interaction, which can be added to the strong force, to extend the gauge group $SU(3)$ acting on $3 + 3$ colours and anti-colours to G_2 acting on 7 'colours'. This extension is reminiscent of the Pati–Salam model, which uses $SU(4)$ for four colours and four anti-colours, but is group-theoretically completely different. Notice also that we have a chiral pair of left-handed and right-handed $SL_2(\mathbb{R})$, so that there is a close parallel between the two models:

$$\begin{aligned} &SU(4) \times SU(2)_L \times SU(2)_R \\ &G_2 \times SL_2(\mathbb{R})_L \times SL_2(\mathbb{R})_R. \end{aligned} \tag{12}$$

We have a total of 20 degrees of freedom, compared to 21 in the Pati–Salam model. We do not use $SL_2(\mathbb{R})_R$, because it does not commute with $Sp_6(\mathbb{R})$. Therefore our full model is of type $C_3 + A_1 + G_2$, based on the group

$$Sp_6(\mathbb{R}) \times SL_2(\mathbb{R}) \times G_2, \tag{13}$$

with the first factor generalising the Lorentz group, the second factor representing a real form of weak $SU(2)$, and the third factor generalizing strong $SU(3)$.

5. Extending to A_5

The occurrence of G_2 in the decomposition $A_1+G_2+C_3$, rather than A_2, that we would expect for the strong force, suggests that we should move to the second group in the third row of the magic square, of type A_5, and the associated decomposition $A_1 + A_2 + A_5$ of E_8. The particular real forms that arise are

$$SL_2(\mathbb{R}) \times SU(3) \times SU(3,3), \tag{14}$$

which is analogous to the decomposition

$$SU(2) \times SL_3(\mathbb{R}) \times SL_3(\mathbb{H}) \tag{15}$$

studied in [1, Section II.C]. I propose these different real forms as a potentially closer match to the Standard Model, since the compact group $SU(3)$ is more suitable for massless gluons, while the split group $SL_2(\mathbb{R})$ has two boosts that are suitable for masses of the intermediate vector bosons. This is because mass is usually introduced by complexifying the compact gauge group, precisely in order to generate boosts.

To extend $Sp_6(\mathbb{R})$ to $SU(3,3)$ we add 14 dimensions, all involving the complex structure l. The analogous 2×2 extension is from $Spin(3,2)$ acting on u, U, K, L, KL to $Spin(4,2)$ acting on l, u, U, K, L, KL, so that the five new elements are

$$X_l, D_l, X_{lK}, X_{lL}, X_{lKL} \tag{16}$$

Since we are not really using the octonions here, all these elements can be written as 2×2 anti-Hermitian matrices over

$$\mathbb{C} \otimes \mathbb{H}' = \langle u, l\rangle \otimes \langle U, K, L, KL\rangle \tag{17}$$

so that the new matrices are

$$\begin{pmatrix} 0 & l \\ l & 0 \end{pmatrix}, \begin{pmatrix} l & 0 \\ 0 & -l \end{pmatrix}, \begin{pmatrix} 0 & lK \\ -lK & 0 \end{pmatrix}, \begin{pmatrix} 0 & lL \\ -lL & 0 \end{pmatrix}, \begin{pmatrix} 0 & lKL \\ -lKL & 0 \end{pmatrix}. \tag{18}$$

To get the whole of $SU(3,3)$, therefore, we need to extend from 2×2 to 3×3 matrices, which means adding in the corresponding Ys and Zs, and one E:

$$\begin{aligned} &E_l, Y_l, Y_{lK}, Y_{lL}, Y_{lKL}, \\ &Z_l, Z_{lK}, Z_{lL}, Z_{lKL}. \end{aligned} \tag{19}$$

The 'diagonal' part of this group is $U(1) \times U(1) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$, an 11-dimensional group generated by

$$D_l, E_l, D_K, E_K, D_{L,KL}, D_L, E_L, D_{K,KL}, D_{KL}, E_{KL}, D_{K,L}, \tag{20}$$

and the off-diagonal part consists of 8 dimensions each of elements of type X (bosonic), Y and Z (fermionic). Adding any one these three types gives a group $U(1) \times SL_2(\mathbb{R}) \times SU(2,2)$ of dimension 19. The subgroup $Sp_6(\mathbb{R})$ loses the first two diagonal elements of type D, and half of the off-diagonal elements of types X, Y, Z. In this case the Ds and Xs generate a group $SL_2(\mathbb{R}) \otimes Sp_4(\mathbb{R})$ of dimension 13.

The particular real form $SU(3,3)$ suggested here as a (huge) generalisation of the Lorentz group, from 6 dimensions to 35, is closely related to Penrose twistors, since the corresponding entry in the magic square of 2×2 matrix groups is $SU(2,2)$. In other words, $SU(3,3)$ combines the group $Sp_6(\mathbb{R})$ of symmetries of phase space with the group $SU(2,2)$ of symmetries of twistors, into a single symmetry group. If this mathematical unification can lead to a physical unification, then it could have far-reaching consequences for the fundamental theory.

It should be noted that $SU(2,2)$ embeds in $Spin(12,4)$ in two different ways, as $Spin(2,4)$ and as $Spin(4,2)$, centralizing $Spin(10)$ and $Spin(8,2)$ respectively. The former was used in [5] and extended to $SU(2,3) \times SU(5)$ in an attempt to understand how twistors relate to E_8 models. Here we use the latter instead, so that the centralizer splits as $Spin(6)\otimes Spin(2,2)$ to give a different real form of the Georgi $Spin(10)$ GUT, and a different embedding of the twistors into E_8. Comparing with [1], we see that the latter uses $Spin(3,3) \otimes Spin(4)$ as yet another real form of $Spin(10)$. It is worthwhile considering which real form is most appropriate. In the Standard Model, the strong force $SU(3)$ is definitely compact, and the mediators are correspondingly massless, but the weak force $SU(2)$ is definitely not compact, because the complexification is used to allow the mediators to be massive. So of the three choices $Spin(10)$, $Spin(7,3)$ and $Spin(8,2)$, only the last has a reasonable chance of agreeing with the Standard Model.

The group $Sp_6(\mathbb{R})$ is studied in detail in [24], embedded in a different real form of $SU(6)$, namely $SL_3(\mathbb{H}')$. It is straightforward to translate that work into $SU(3,3)$, simply by multiplying the Hermitian matrices by l to make them anti-Hermitian over $\mathbb{C}\otimes\mathbb{H}'$. However, any interpretation in [24] that is based on the particular real form is suspect. The study of $SL_3(\mathbb{H})$ in [1] is also not difficult to translate into, or out of, these other two real forms. But again, the interpretation offered in [1] is quite different from the interpretation I offer here. The great advantage of using $SU(3,3)$ rather than $SL_3(\mathbb{H}')$ is that it removes the contradiction with general relativity that was apparent in [24] (see Section 9 below).

6. Representations

Let us first look at the restriction to $A_1 + A_2 + A_5$ of the adjoint representation of E_8. The real constituents for the real form $SL_2(\mathbb{R}) \times SU(3) \times SU(3,3)$ are as follows:

$$\begin{aligned}
\mathbf{3} &= \mathbf{3}\otimes\mathbf{1}\otimes\mathbf{1}\\
\mathbf{8} &= \mathbf{1}\otimes\mathbf{8}\otimes\mathbf{1}\\
\mathbf{35} &= \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{35}\\
\mathbf{40} &= \mathbf{2}\otimes\mathbf{1}\otimes\mathbf{20}\\
\mathbf{90} &= \mathbf{1}\otimes\mathbf{3}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathbf{15}_{\mathbb{C}}\\
\mathbf{72} &= \mathbf{2}\otimes\mathbf{3}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathbf{6}_{\mathbb{C}}
\end{aligned}\tag{21}$$

The first three constituents are the adjoint representations of the three factors, and the last three involve the real weak doublet representation $\mathbf{2}$ and the complex colour triplet representation $\mathbf{3}_{\mathbb{C}}$. The representations of $SU(3,3)$ are the natural 6-dimensional complex representation $\mathbf{6}_{\mathbb{C}}$, its anti-symmetric square $\mathbf{15}_{\mathbb{C}}$ and its anti-symmetric cube (real $\mathbf{20}$).

Restricting to $SU(2,2)$ to separate bosonic and fermionic representations we have

$$\begin{aligned}\mathbf{6}_{\mathbb{C}} &\to (1+1)+4\\ \mathbf{20} &\to (6+6)+(4+4)\\ \mathbf{15}_{\mathbb{C}} &\to (1+6)+(4+4)\end{aligned} \tag{22}$$

which gives us a total of $16+48+48=112$ dimensions of spinors, of which 48 are right-handed and 64 are left-handed. The left-handed spinors split $16+48$ into leptons and quarks, while the right-handed spinors here represent only quarks. The remaining 16 dimensions of right-handed spinors lie inside the group $SU(3,3)$. In order to allocate some of these spinors to right-handed electrons, therefore, we need to break the symmetry back down to $Sp_6(\mathbb{R})$. This gives us 8 dimensions of right-handed lepton spinors, compared to 16 for left-handed leptons, which is the ratio that we expect.

One way to study the splitting of E_8 into representations of $A_1+C_3+G_2$ is to embed it first in F_4+G_2, where we have a decomposition

$$\mathbf{248} = \mathbf{52}+\mathbf{14}+\mathbf{26}\otimes\mathbf{7}. \tag{23}$$

Then we restrict from F_4 to A_1+C_3 to get

$$\begin{aligned}\mathbf{52} &= \mathbf{3}+\mathbf{21}+\mathbf{2}\otimes\mathbf{14}b\\ \mathbf{26} &= \mathbf{2}\otimes\mathbf{6}+\mathbf{14}a\end{aligned} \tag{24}$$

where $\mathbf{6}$ is the natural representation of $Sp_6(\mathbb{R})$, and the other representations are defined by

$$\begin{aligned}\Lambda^2(\mathbf{6}) &= \mathbf{1}+\mathbf{14}a,\\ S^2(\mathbf{6}) &= \mathbf{21},\\ \Lambda^3(\mathbf{6}) &= \mathbf{6}+\mathbf{14}b.\end{aligned} \tag{25}$$

Here, $\Lambda^2(\mathbf{6})$ has a natural structure as a Jordan algebra, while $S^2(\mathbf{6})$ has a natural structure as a Lie algebra. An alternative way to see these splittings is by restriction from $SU(3,3)$:

$$\begin{aligned}\mathbf{6}_{\mathbb{C}} &\to \mathbf{6}\\ \mathbf{15}_{\mathbb{C}} &\to \mathbf{1}+\mathbf{14}a\\ \mathbf{20} &\to \mathbf{6}+\mathbf{14}b\\ \mathbf{35} &\to \mathbf{21}+\mathbf{14}a\end{aligned} \tag{26}$$

As representations of $A_1+G_2+C_3$ we have the following irreducible constituents of E_8:

$$\begin{aligned}\mathbf{3} &= \mathbf{3}\otimes\mathbf{1}\otimes\mathbf{1}\\ \mathbf{14} &= \mathbf{1}\otimes\mathbf{14}\otimes\mathbf{1}\\ \mathbf{21} &= \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{21}\\ \mathbf{28} &= \mathbf{2}\otimes\mathbf{1}\otimes\mathbf{14}b\\ \mathbf{98} &= \mathbf{1}\otimes\mathbf{7}\otimes\mathbf{14}a\\ \mathbf{84} &= \mathbf{2}\otimes\mathbf{7}\otimes\mathbf{6}\end{aligned} \tag{27}$$

All the right-handed spinors (including the electrons) are now inside the **98**, while the left-handed lepton spinors have been split $8+8$ between the **28** and the **84**, as a result of the breaking of **20** into $\mathbf{14}b+\mathbf{6}$. This curious phenomenon will no doubt repay closer scrutiny. It looks at first sight like a distinction between (massless) neutrinos and (massive) electrons, but that is not consistent with the identification of the A_1 factor as acting on weak doublets. Hence one or other of these suggested interpretations has to change.

This analysis gives us a total of five different types of spinors:

- 8 dof in 28, left-handed leptons;
- $8+48$ dof in 98, right-handed leptons and quarks;
- $8+48$ dof in 84, left-handed leptons and quarks.

In total, then, there are 24 dof for leptons, or 6 Weyl spinors, compared to the 9 that are usually expected for three generations. Similarly, there are 96 dof for quarks, or 24 Weyl spinors, compared to the 36 that are usually expected. Thus we need a mechanism similar to that proposed in [1] for reducing the number of independent spinors required by one-third. This is not surprising, of course, as it is well-known that the standard interpretation requires 180 dof for spinors [6]. However, a *discrete* symmetry of order 3 can be implemented in a 2-dimensional real space using the symmetries of an equilateral triangle, so there is no theoretical reason why a 3-space is needed for this symmetry.

7. Restricting to $A_2 + A_2$

An alternative strategy for producing spinors for right-handed electrons is to restrict from A_5 to $A_2 \times A_2$ instead of C_3. This extends the centralizer from $A_1 + A_2$ to $A_2 + A_2$, and gives rise to the following subgroup of $E_{8(-24)}$:

$$SL_3(\mathbb{C}) \times SU(3) \times SL_3(\mathbb{R}). \tag{28}$$

This group provides an obvious embedding of the Lorentz group $SL_2(\mathbb{C})$ in $SL_3(\mathbb{C})$, defining the splitting into fermions and bosons, and offers various possibilities for $SL_2(\mathbb{R})$ or $SO(3)$ for the weak force. However, $SL_3(\mathbb{R})$ acts identically on vectors and both types of spinors, so does not provide any obvious way to implement the chirality of the weak force. We considered this possibility in the work that led to [1], but did not find a way to make it work. Of course, that does not necessarily mean that it cannot be done.

The group $SL_3(\mathbb{C})$ is generated by the 16 elements

$$\begin{aligned} &D_l, D_L, X_1, X_l, X_L, X_{lL},\\ &E_l, E_L, Y_1, Y_l, Y_L, Y_{lL},\\ &Z_1, Z_l, Z_L, Z_{lL} \end{aligned} \tag{29}$$

in which the top row is the Lorentz group $SL_2(\mathbb{C})$, centralized by a complex scalar generated by

$$\begin{aligned} E_L - F_L &= D_L + 2E_L,\\ E_l - F_l &= D_l + 2E_l. \end{aligned} \tag{30}$$

The group $SU(3)$ acts on the labels i, j, k, il, jl, kl, identically on X, Y and Z. The group $SL_3(\mathbb{R})$ acts similarly on the labels I, J, K, IL, JL, KL. The 120 spinors therefore split as $24+24+72$, in which the 72 have both an i, j, k and an I, J, K in the label, and the 24s have one or the other.

We now have to break the symmetry of I, J, K in order to separate left-handed and right-handed spinors, so let us separate I, J from K to break $72 = 24 + 48$ and one of the $24 = 8 + 16$. This gives us a splitting of quarks in $24 + 24 + 48$, such that $24 + 24$ are right-handed, and 48 are left-handed. Similarly, the leptons split as 8 right-handed and 16 left-handed, again in agreement with the Standard Model. To be more explicit, we give the labels in the form of a table, with the labels for the C_3 version for comparison:

$$\begin{array}{r|cc|cc|}
\text{RH l} & u,l & K,KL & u,l & U\pm KL, K\pm L \\
\text{LH l} & u,l & I,J,IL,JL & u,l & I,J,IL,JL \\
\text{RH q} & i,j,k,il,jl,kl & U,K,L,KL & i,j,k,il,jl,kl & U,K,L,KL \\
\text{LH q} & i,j,k,il,jl,kl & I,J,IL,JL & i,j,k,il,jl,kl & I,J,IL,JL
\end{array} \tag{31}$$

The actual signs for the right-handed leptons in the C_3 case are different in the Y and Z spinors, but only one sign occurs in each case. The allocation of individual particles in the C_3 and $A_2 + A_2$ cases is not necessarily the same, but the overall picture is very similar. But only the C_3 case has the projection with $U - KL$ that corresponds to $1 - \gamma_5$ in the Standard Model. This appears to be a decisive vote in favour of the C_3 model over the $A_2 + A_2$ model.

8. Symmetry-breaking

Nevertheless, there are many questions remaining about the differences between C_3 and A_2+A_2, particularly concerning the physical interpretations, and the reason for the symmetry-breaking from $SL_3(\mathbb{R})$ to $SL_2(\mathbb{R})$. In order to bring the questions into focus, it is useful to consider the square of groups A_2, $A_2 + A_2$, C_3 and A_5, together with the corresponding block of the 2×2 magic square, and the centralizers.

$$\begin{array}{ccc}
\boxed{\begin{array}{cc} SL_3(\mathbb{R}) & SL_3(\mathbb{C}) \\ Sp_6(\mathbb{R}) & SU(3,3) \end{array}} & & \boxed{\begin{array}{cc} Spin(2,1) & Spin(3,1) \\ Spin(3,2) & Spin(4,2) \end{array}} \\
 & \boxed{\begin{array}{cc} G_2\times SL_3(\mathbb{R}) & SU(3)\times SL_3(\mathbb{R}) \\ G_2\times SL_2(\mathbb{R}) & SU(3)\times SL_2(\mathbb{R}) \end{array}} &
\end{array} \tag{32}$$

We would expect to use $Spin(3,1)$ in the top right corner for the Lorentz group, acting on the four labels u, U, l, L for spacetime and/or 4-momentum. However, in the Dirac algebra including γ_5 we need $Spin(3,2)$, which appears in the bottom-left, acting on the five labels u, U, K, L, KL. From this labelling we see that we have lost one of the three dimensions of momentum, labelled l, and gained two dimensions of something else, labelled K, KL. In particular, we have broken the symmetry of spacetime, to include a preferred direction in space, and we have broken the symmetry of I, J, K, to obtain a weak force with a broken symmetry group $SL_2(\mathbb{R})$. These two types of symmetry-breaking are independent of each other, but both are needed to obtain the Standard Model of electro-weak interactions.

The breaking of the spacetime-symmetry is usually disregarded, and simply expressed as a choice of the z direction in which to measure spin. But in the E_8 model it seems more likely that it is something much more important than that, such as the direction of acceleration relative to an inertial frame, or the direction of the ambient gravitational field, or the direction of the ambient angular momentum. At any rate, it must be a direction that has physical meaning and can be measured. The breaking of the I, J, K symmetry seems most likely to be a breaking of the generation symmetry of fundamental fermions.

This suggests that the top right corner, in the form

$$SL_3(\mathbb{C}) \times SU(3) \times SL_3(\mathbb{R}), \tag{33}$$

represents the Standard Model of elementary particles, with three colours of quarks and three generations of fermions (but only for half of the right-handed quarks!). The restriction from $SL_3(\mathbb{C})$ to $SL_2(\mathbb{C})$ splits fermions from bosons, and allows every observer to choose their own preferred copy of the Lorentz group $SL_2(\mathbb{C})$. The ten remaining dimensions of $SL_3(\mathbb{C})$ consist of a complex scalar and a Dirac spinor. Hence there is an 8-parameter family of copies of $SL_2(\mathbb{C})$ available for different observers. This compares to a 9-parameter family of copies of $SO(3,1)$ inside $SL_4(\mathbb{R})$ that describes the analogous phenomenon in General Relativity. Clearly, therefore, this model does not resolve the basic problem of incompatibility of GR with QM. It does, however, provide an explanation of sorts for the so-called 'right-handed neutrinos': these are not interpreted as particles, but as transformations between different coordinate systems preferred by different observers.

The interpretation of the bottom left corner, in the form

$$Sp_6(\mathbb{R}) \times G_2 \times SL_2(\mathbb{R}), \tag{34}$$

is now freed from the necessity to include the Lorentz group in $Sp_6(\mathbb{R})$, and hence freed from the necessity to extend Minkowski spacetime, with symmetry group $SO(3,1)$, to anti-de Sitter spacetime, with symmetry group $SO(3,2)$. Indeed, the latter group, or rather its double cover $Sp_4(\mathbb{R})$, now only has to act on two of the three spatial coordinates, and can therefore plausibly be identified with the group of symmetries of phase space for 2-dimensional dynamics. The choice of which two dimensions these are is the same as the choice of restriction from $SO(3,1)$ to $SO(2,1)$ above, and therefore has the same relationship to acceleration, rotation and/or the gravitational field. In practice, most experiments are horizontal, so that the most likely direction in most cases will be the direction of the gravitational field. However, other directions may enter into the model at various points.

Finally, the extension from $Sp_4(\mathbb{R})$ to $Sp_6(\mathbb{R})$ allows individual observers to choose whichever 2-dimensional part of dynamics they wish to model with the Standard Model, and hence to modify the Standard Model for different ambient conditions of acceleration, rotation and gravity. If we want a model that is truly relativistic (independent of the observer), then we can treat $Sp_6(\mathbb{R})$ as the symmetry group of phase space for 3-dimensional dynamics, which entails abandoning the concept of 'spacetime', since it is now observer-dependent, and therefore no longer useful for a fundamental theory. The embedding of $Sp_4(\mathbb{R})$ via $Sp_4(\mathbb{R}) \times Sp_2(\mathbb{R})$ into $Sp_6(\mathbb{R})$ shows that there is again an 8-parameter family of copies of $Sp_4(\mathbb{R})$ available, for an 8-parameter family of observers. This extends to an 11-parameter family of copies of $SL_2(\mathbb{C})$, in case this is useful.

A more radical proposal is made in [24], in which it is noted that both $SU(3)$ and $SL_3(\mathbb{R})$ are subgroups of $Sp_6(\mathbb{R})$, and it is therefore proposed to identify the colour symmetry group $SU(3)$ and the generation symmetry group $SL_3(\mathbb{R})$ with the corresponding subgroups of $Sp_6(\mathbb{R})$, so that the latter group now contains all of the required symmetries of fundamental physics. It may be that this proposal is too radical, but it is certainly necessary to have *some* mechanism for linking the generation symmetry group $SL_3(\mathbb{R})$ to *some* concept of mass.

9. Gravity

It must be stressed that with the standard interpretation of the Lorentz group $SL_2(\mathbb{C})$ acting on spacetime, labelled by u, U, l, L, there is no conceivable way of implementing General Relativity inside $E_{8(-24)}$. The fact that $SL_4(\mathbb{R})$ has no double cover that acts on spinors categorically rules this out. However, there are bits of E_8 that appear not to be used in the Standard Model, that could in principle be used for a quantum theory of gravity, and that might approximate to GR in appropriate circumstances. Or it may be that a suitable tweak to the interpretation might allow GR in to the model.

For example, $SL_4(\mathbb{R})$ acting on spacetime translates to $SO(3,3)$ acting on phase space, and there is an obvious copy of $SO(3,3)$ in $SU(3,3)$, that extends $SL_3(\mathbb{R})$ in $Sp_6(\mathbb{R})$. This suggests that complexifying phase space, in order to implement momentum and current separately, may be key to including quantum gravity in the model. In other words, A_5 is the battleground in which the QM and GR models must resolve their differences. These differences mainly lie in the fundamental properties of spacetime, so the hope is that by shifting the emphasis from spacetime to phase space, a compromise might be reached in which spacetime is never defined or used at all. If QM uses $Sp_6(\mathbb{R})$ to act on phase space, as it should in a Hamiltonian theory, and GR uses $SO(3,3)$, then both can be embedded in $SU(3,3)$, and each can be allowed to add corrections to the other.

The key to this process is to re-interpret the various spin groups as acting on 2+2-dimensional (real, complex or quaternionic) phase space, and not on an abstract space of 'spinors' that have no concrete physical interpretation. We have already done this with the group $Spin(3,2) \cong Sp_4(\mathbb{R})$ in C_3, interpreted as a subalgebra of the Dirac algebra, so we must now do the same with $Spin(4,2) \cong SU(2,2)$ in A_5, and interpret this both as the full complex Dirac algebra, and as a complex version of the subgroup $SO(2,2)$ of $SO(3,3)$, corresponding to a subgroup $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$ of $SL_4(\mathbb{R})$ in GR.

Within the 2×2 magic square, we therefore have a copy of

$$SO(2,2) = Spin(2,1) \otimes Spin(2,1) \tag{35}$$

inside $Spin(4,2)$, from which we can write down generators for $SO(3,3)$ as follows:

$$\begin{array}{r} D_L, D_{K,KL}, X_1, X_L, X_{lK}, X_{lKL}, \\ E_L, Y_1, Y_L, Y_{lK}, Y_{lKL}, \\ Z_1, Z_L, Z_{lK}, Z_{lKL} \end{array} \tag{36}$$

The first row consists of generators for $SO(2,2)$. The elements whose labels do not include l generate $GL_3(\mathbb{R})$ inside $Sp_6(\mathbb{R})$. This choice of $SO(2,2)$ amounts to splitting the 6 labels into u, U, L for one copy of $Spin(2,1)$ and l, K, KL for the other. In matrix terms we have

$$\begin{array}{l} \begin{pmatrix} L & \\ 0 & -L \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & L \\ L & 0 \end{pmatrix}, \\ \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}, \begin{pmatrix} 0 & lK \\ -lK & 0 \end{pmatrix}, \begin{pmatrix} 0 & lKL \\ -lKL & 0 \end{pmatrix}, \end{array} \tag{37}$$

where the first row is $SO(2,1)$ acting on $2+1$-dimensional spacetime, and the second row acts on $2+1$ particles.

This splitting is similar to Woit's splitting [23] into 'right-handed' spacetime and 'left-handed' gauge groups, respectively, except that he complexifies everything and interprets these groups as compact $SU(2)$ rather than split $SL_2(\mathbb{R})$. In our interpretation, the 'right-handed' copy acts on 2+1-spacetime, and the 'left-handed' copy acts on $2+1$ fermions in a single generation. At this point we may notice that we have acquired two copies of $SL_2(\mathbb{R})$, both labelled 'left-handed', one acting on labels l, K, KL, the other acting on labels I, J, IL, JL. Some mixing of the two may be required in order to define masses for the intermediate vector bosons.

The tensors used in GR are irreducible representations of $SO(3,3)$ of dimensions 6 (field strength tensor), 10 (Ricci tensor, stress-energy tensor) and 20 (Riemann curvature tensor). The representations of $SU(3,3)$ that are available in E_8 are complex $\mathbf{6}_\mathbb{C}$ and $\mathbf{15}_\mathbb{C}$, and real $\mathbf{20}$ and $\mathbf{35}$, which restrict to $SO(3,3)$ as follows:

$$\begin{aligned} \mathbf{6}_\mathbb{C} &\to \mathbf{6} \\ \mathbf{15}_\mathbb{C} &\to \mathbf{15} \\ \mathbf{20} &\to \mathbf{10}a + \mathbf{10}b \\ \mathbf{35} &\to \mathbf{15} + \mathbf{20} \end{aligned} \tag{38}$$

It follows that the Einstein field equations must be written in terms of the representation $\mathbf{2} \otimes \mathbf{20}$ on left-handed leptons, that relates matter to neutrinos, and therefore expresses the gravitational field in terms of neutrinos. On the other hand, the Riemann Curvature Tensor(RCT) lies in the adjoint representation of $SU(3,3)$, and consists of the 20 dimensions outside $SO(3,3)$. Thus the RCT becomes part of the symmetry group of the model. Finally, the field strength tensor is coupled to the quark colour/charge representation $\mathbf{2} \otimes \mathbf{3}_\mathbb{C}$, and hence to baryonic matter.

From this analysis, we see that the only essential thing that is missing from GR is the distinction between $\mathbf{10}a$ and $\mathbf{10}b$, which are usually regarded as being self-dual and therefore equivalent, but are in fact dual *to each other*. In the usual formalism in terms of the Lorentz group $SO(3,1)$, they both restrict to $\mathbf{1} + \mathbf{9}$, so that the distinction between them is less obvious. The introduction of a second scalar (the cosmological constant) only extends from 10 to 11 variables, when 20 are required for the full theory. A consequence of this generalisation is that electrons are far more important for gravity than is usually supposed, since all three generations of neutrinos and antineutrinos, and therefore all three generations of electrons, participate in an essential way.

10. Conclusion

The problem of unification of particle physics and gravity goes back almost a century, and occupied Einstein for at least a quarter of that century. Yet the problem seems no nearer to a solution today than it did fifty years ago, and even further away than it seemed forty or thirty years ago. This indicates that there must be something subtly wrong in the basic assumptions somewhere. My analysis locates this problem in the concept of spacetime itself. The way that spacetime is treated in relativity, using the Lorentz group in the form $SO(3,1)$, is *mathematically* (never mind physically) inconsistent with the way that spacetime is treated in quantum mechanics, using the Lorentz group in the form $SL_2(\mathbb{C})$. It isn't a question of one of them being 'right' and the other one 'wrong', it is a question of there being no consistent definition of spacetime at all, and no possible way to measure spacetime in the absence of objects embedded in spacetime.

Therefore I have considered the possibility of describing physics without using spacetime, but instead using only phase space, as Hamilton taught us to do. This involves re-interpreting the Dirac algebra, the Einstein field equations and the Riemann curvature tensor in terms of a complex phase space, in order to include both momentum and current. Taking my cue from the variety of models based on E_8 and the magic square, from $E_8 \times E_8$ heterotic string theory down, I focus on the group $SU(3,3)$ embedded in $E_8(-24)$, and find within it all the mathematical structures that a unified model requires.

This is not, of course, in itself a unified theory of fundamental physics. But it is a unified mathematical model, in which all the ingredients of all the fundamental theories of physics can be found. This includes the complex Dirac algebra, the gauge groups of the weak and strong nuclear forces, including symmetry-breaking of the weak force, a classification of elementary fermions in which there are no right-handed neutrinos, general covariance, and all the tensors used in GR. Moreover, these ingredients fit together in ways that are broadly consistent with experiment. I therefore suggest that this is a promising foundation on which to try to build a unified theory of fundamental physics.

References

[1] C. A. Manogue, T. Dray and R. A. Wilson (2022), Octions: an E_8 description of the standard model, *J. Math. Phys.* **63**, 081703. arXiv:2204.05310

[2] A. G. Lisi (2007), An exceptionally simple theory of everything, arXiv:0711.0770

[3] A. G. Lisi (2010), An explicit embedding of gravity and the standard model in E8, arXiv:1006.4908

[4] D. Chester, A. Marrani and M. Rios (2020), Beyond the standard model with six-dimensional spacetime, arXiv:2002.02391.

[5] R. A. Wilson (2022), Chirality in an E_8 model of elementary particles, arXiv:2210.06029.

[6] J. Distler and S. Garibaldi (2010), There is no E_8 theory of everything, *Communications in Math. Phys.* **298** (2), 419–436.

[7] H. Freudenthal, Beziehungen der E_7 und E_8 zur Oktavenebene. X-XI. *Indag. Math.* **25** (1963), 457–487.

[8] J. Tits, Algèbres alternatives, algèbres de Jordan et algèbres de Lie exceptionelles, *Indag. Math.* **28** (1966), 223–237.

[9] C. H. Barton and A. Sudbery, Magic squares and matrix models of Lie algebras, *Advances in Math.* **180** (2003), 596–647.

[10] C. A. Manogue and T. Dray (2010), Octonions, E6, and particle physics, *J. Phys. Conf. Ser.* **254**, 012005.

[11] I. Todorov and M. Dubois-Violette (2018), Deducing the symmetry of the standard model from the automorphism and structure groups of the exceptional Jordan algebra, *Int. J. Mod. Phys. A* **33**, 1850118

[12] J. C. Pati and A. Salam (1974), Lepton number as the fourth 'color', *Phys. Rev. D* **10** (1), 275–289.

[13] R. Penrose (1967), Twistor algebra, *J. Math. Phys.* **8** (2), 345–366.

[14] T. Adamo (2017), *Lectures on twistor theory.* arXiv:1712.02196.

[15] J. Maldecena (1998), The large N limit of superconformal field theories and supergravity, *Adv. Theor. Math. Phys.* **2**, 231–252.

[16] T. Dray. C. A. Manogue and R. A. Wilson (2014), A symplectic representation of E_7, *Comment. Math. Univ. Carolin.* **55**, 387.

[17] T. Dray, C. A. Manogue and R. A. Wilson (2024), A new division algebra representation of E_7, arXiv:2401.105

[18] M. de Gosson and B. Hiley (2011), Imprints of the quantum world in classical mechanics, *Found. Phys.* **41**, 1415–1436.

[19] R. A. Wilson, T. Dray and C. A. Manogue (2022), An octonionic construction of E_8 and the Lie algebra magic square, arXiv:2204.04996

[20] M. Gell-Mann (1961), The eightfold way: a theory of strong interaction symmetry, Synchrotron Lab. Report CTSL-20, Cal. Tech.
[21] W. Greiner, S. Schramm and E. Stein (2007), *Quantum Chromodynamics*, Springer.
[22] S. Coleman and J. Mandula (1967), All possible symmetries of the S matrix, *Physical Review* **159** (5), 1251.
[23] P. Woit (2021), Euclidean spinors and twistor unification, arXiv:2104.05099
[24] R. A. Wilson (2024), A Clifford algebra model in phase space, arXiv:2404.04278.

Queen Mary University of London
Email address: r.a.wilson@qmul.ac.uk